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On Multiplication Groups of Loops

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1. INTRODUCTION

The concept of multiplication groups of quasigroups was introduced by Albert [1] and the connection between quasigroups and corresponding multiplication groups has been studied by Bruck [6], Smith [20] and Ihringer [14, 15].

While studying the multiplication group of a loop Q (a quasigroup with neutral element) a central role is played by the stabilizer of the neutral element. This subgroup $I(Q)$ of the multiplication group is called the inner mapping group of Q. If Q is a group then it is clear that $I(Q)$ consists of the inner automorphisms of Q. We also know that a loop Q is an abelian group if and only if $I(Q) = 1$.

In this paper we study some properties of the inner mapping group and we also give a partial answer to the question: What are the multiplication groups of loops? This question is closely connected to certain transversal conditions. Sections 2 and 3 are devoted to investigating these conditions and in Section 4 we characterize multiplication groups of loops with the aid of these conditions. In the same section we prove one of our main results: If Q is a finite loop whose inner mapping group is cyclic, then Q is an abelian group. Finally, in Section 5 we use the properties of the inner mapping group in order to show that certain groups are not multiplication groups of loops. We also give examples of groups which are multiplication groups of loops.

Our notation is standard and for basic facts about groups and loops we refer to [4, 7, 13].

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2. Connected Transversals

Let H be a subgroup of G. We say that a left transversal A to H in G is *stable* if Ax is a left transversal to H in G for every $x \in G$. Stable transversals are sometimes also called loop transversals. They are discussed, e.g., in [3, 15, 16]. It is not difficult to prove

Lemma 2.1. *The following conditions are equivalent:*

(1) *A is a stable transversal to H in G,*

(2) *A^g is a left transversal to H in G for every $g \in G$,*

(3) *A^g is a right transversal to H in G for every $g \in G$.*

Let A and B be two left transversals to H in G. We say that A and B are *H-connected* if $[A, B] \leqslant H$. If A is a left transversal to H in G and $[A, A] \leqslant H$, then we say that A is an *H-selfconnected transversal.*

Now we denote by $L_G(H)$ the core of H in G. Suppose that A and B are H-connected and $L_G(H) = 1$. If $x \in A \cap H$, then $b^{-1}x^{-1}bx \in H$ for every $b \in B$. But then $x \in L_G(H) = 1$, hence $1 \in A$. In the same way we can show that $1 \in B$. Now it is clear that if A is an H-selfconnected transversal and $L_G(H) = 1$, then $1 \in A$.

Lemma 2.2. *Let A and B be H-connected transversals in G. Then A and B are both stable transversals to H in G.*

Proof. Consider the set Ax, where $x \in G$. Let $a, c \in A$ and $x = bh$, where $b \in B$ and $h \in H$. If $(ax)^{-1}(cx) \in H$, then $h^{-1}b^{-1}a^{-1}cbh \in H$ and $b^{-1}a^{-1}cb \in H$. Now

$$a^{-1}c = (a^{-1}b^{-1}ab)(b^{-1}a^{-1}cb)(b^{-1}c^{-1}bc),$$

hence $a^{-1}c \in H$ and we conclude that $a = c$ and $ax = cx$. Now let $y \in G$. If $x = bh$ as before and if $b^{-1}y = dk$ $(d \in A,\ k \in H)$, then $y = (dx)l$, where $l = h^{-1}b^{-1}d^{-1}bdk \in H$. Thus Ax is a left transversal and A is stable. In a similar way it can be shown that B is stable.

Example 2.3. Let K be an arbitrary nonabelian group and $H = \mathrm{Inn}(K)$, the group of inner automorphisms of K. Let $G = HK$ be the semidirect product of H and K. For $k \in K$, let $\bar{k}$ denote the inner automorphism $x \to k^{-1}xk$. Let $A = K$ and $B = \{\bar{k}k^{-1} \mid k \in K\}$. Now A and B are left transversals to H in G and $A^g = A$, since A is normal in G. If $l \in A$ and $\bar{k}k^{-1} \in B$, then $l\bar{k}k^{-1} = \bar{k}k^{-1}l$ and thus $[A, B] \leqslant H$. (We are indebted to Stephen Glasby who pointed out this example.)

EXAMPLE 2.4. Let $G=A_4$ and $H=\{e,(12)(34)\}$. Then there exist no stable transversals to H in G.

LEMMA 2.5. *Let $H \leqslant G$ and A and B be H-connected transversals. Let $C \subseteq A \cup B$ and $K=\langle H, C\rangle$. Then $C \subseteq L_G(K)$.*

Proof. We must prove that $x^{-1}cx \in K$ for every $c \in C$ and for every $x \in G$. We can assume that $c \in A$ and $x=bu$ for some $b \in B$ and $u \in H$. Now $x^{-1}c^{-1}x=u^{-1}b^{-1}c^{-1}bu=u^{-1}b^{-1}c^{-1}bcc^{-1}u$. Since $b^{-1}c^{-1}bc \in H$, it follows that $x^{-1}c^{-1}x \in K$.

LEMMA 2.6. *Let H be a proper subgroup of a simple group G and let A and B be H-connected transversals in G. Then H is maximal in G.*

Proof. Let $a \in A-H$ and put $K=\langle a, H\rangle$. By Lemma 2.5, $a \in L_G(K)$. Since $L_G(K) \neq 1$ and G is simple, we conclude that $L_G(K)=G$. But then $K=G$ and thus H is a maximal subgroup of G.

PROPOSITION 2.7. *Let $H \leqslant G$ and suppose that A and B are H-connected transversals in G. If $L_G(H)=1$, then $N_G(H)=H \times Z(G)$.*

Proof. For every $x \in N_G(H)$ we can define a mapping $f_x\colon A \to A$ by $x^{-1}ax \in f_x(a)H$. By Lemmas 2.1 and 2.2, f_x is well-defined. Clearly, f_x is a permutation on A. If $x, y \in N_G(H)$ and $a \in A$, then $x^{-1}ax=f_x(a)u$, $y^{-1}f_x(a)y=f_y(f_x(a))v$, and $y^{-1}x^{-1}axy=f_{xy}(a)w$ for some u, v and $w \in H$. Then $f_{xy}(a)w=f_y(f_x(a))vy^{-1}uy$ and since $vy^{-1}uy \in H$, it follows that $f_{xy}(a)=f_yf_x(a)$. Thus the mapping $F\colon x \to f_{x^{-1}}$ is a group homomorphism from $N_G(H)$ to S_A (the symmetric group on A). Since A and B are H-connected, we have $D=B \cap N_G(H) \subseteq \mathrm{Ker}(F)$. Now we write $L=\mathrm{Ker}(F) \cap H$. Clearly, if $x \in L$, then $x \in H$ and $a^{-1}xa \in H$ for every $a \in A$. Hence $L \subseteq L_G(H)=1$.

Now $N_G(H)=DH$ and since $L=1$, we have $\mathrm{Ker}(F)=\langle D\rangle=D$. Moreover, D is normal in $N_G(H)$; hence $N_G(H)=H \times D$.

It remains to show that $D \leqslant Z(G)$. If $g \in G$, then $g=ah$, where $a \in A$ and $h \in H$. If $d \in D$, then $d^{-1}g^{-1}dg=d^{-1}h^{-1}a^{-1}dah=h^{-1}d^{-1}a^{-1}dah \in H$. Thus $E=[D, G] \leqslant H$ and since E is normal in G, it follows that $E=1$. Therefore $D \leqslant Z(G)$ and $N_G(H)=H \times Z(G)$.

LEMMA 2.8. *Let $H \leqslant G$ and let A and B be H-connected transversals in G. Consider a normal subgroup N of G, put $L=L_G(HN)$, and denote by f the natural homomorphism of G onto G/L. Then $f(A)$ and $f(B)$ are $f(H)$-connected transversals in G/L.*

Proof. Let $a, c \in A$ such that $f(a^{-1}c) \in f(H)$. Then $f(b^{-1}a^{-1}cb)=$

$f(b^{-1}a^{-1}ba \cdot a^{-1}b^{-1}bc \cdot c^{-1}b^{-1}cb) \in f(H)$ for every $b \in B$. From this, $f(a^{-1}c) \in L_{G/L}(f(H)) = 1$ and $f(a) = f(c)$. For $f(B)$ we can proceed in a similar way.

3. Two Theorems

In our analysis of the relationship between a subgroup H and H-connected transversals A and B we shall examine the situation where H is cyclic and $L_G(H) = 1$. We first prove

THEOREM 3.1. *Let H be a subgroup of G such that $L_G(H) = 1$ and suppose that H is either a cyclic p-group or isomorphic to the Prüfer-group C_{p^∞} for a prime p. Let A and B be H-connected transversals. Then $A = B$ is an abelian subgroup of G.*

Proof. For every $a \in A$ there exists $f(a) \in B$ with $aH = f(a)H$. Hence $f(a)^{-1}a \in H$ and $f(a)^{-1}aaH = f(a)^{-1}af(a)H = f(a)^{-1}af(a)f(a)^{-1}a^{-1}f(a)aH = aH$. Now we write $c = f(d)^{-1}d$ for a fixed $d \in A$. Let $b \in A$ be arbitrary and let K denote the subgroup generated by c and $f(b)^{-1}b$. Clearly, K is a cyclic p-group, hence K is generated either by c or by $f(b)^{-1}b$. If $f(b)^{-1}b$ is a generator, then $cbH = (f(b)^{-1}b)^n bH = bH$ and we get $b^{-1}cb \in H$.

Now assume that c is a generator of K. Then $f(b)^{-1}bdH = dH = f(d)H$. Now $d^{-1}f(b)^{-1}bd \in H$ and thus

$$b^{-1}d^{-1}bd = b^{-1}f(b) \cdot f(b)^{-1}d^{-1}f(b)d \cdot d^{-1}f(b)^{-1}bd \in H.$$

Moreover, $f(d)^{-1}f(b)^{-1}bd \in H$ and consequently

$$\begin{aligned} &f(b)^{-1}f(d)^{-1}bd \\ &\quad = f(b)^{-1}b \cdot b^{-1}f(d)^{-1}bf(d) \cdot f(d)^{-1}f(b)^{-1}bd \cdot d^{-1}b^{-1}f(b)d \in H. \end{aligned}$$

Now $f(b)^{-1}cb = f(b)^{-1}f(d)^{-1}bdd^{-1}b^{-1}db$ belongs to H and finally $b^{-1}cb = b^{-1}f(b)f(b)^{-1}cb \in H$.

We have shown that in both cases $b^{-1}cb \in H$ and we conclude that $c \in L_G(H) = 1$; hence $f(d) = d$. But this means that $A = B$. Since A is H-selfconnected, we have $abH = baH$ for all $a, b \in A$.

Now we shall show that A is a subgroup of G. For every pair $(a, b) \in A \times A$ there exists a unique $g(a, b) \in A$ such that $abH = g(a, b)H$. Now $g(a, b) = g(b, a)$ and if we write $h(a, b) = g(a, b)^{-1}ab$, then $h(a, b) \in H$ and $h(a, b)b^{-1}a^{-1}ba = h(b, a)$. Moreover, $h(a, b)aH = g(a, b)^{-1}abaH = g(a, b)^{-1}aabH = g(a, b)^{-1}ag(a, b)H = aH$.

Let a, b, and c belong to A and denote by R the subgroup generated by

$h(a, b)$, $h(c, b)$, and $h(c, c)$. Again, R is generated by one of these elements. If R is generated by $h(c, b)$ (or $h(c, c)$), then $h(a, b)cH = cH$ and thus $c^{-1}h(a, b)c \in H$.

Next assume that $h(a, b)$ is a generator of R. Then $u = a^{-1}h(c, b)a \in H$, $v = g(c, b)^{-1}a^{-1}cba = g(c, b)^{-1}a^{-1}g(c, b)au \in H$, $w = b^{-1}c^{-1}a^{-1}cba = h(c, b)^{-1}v \in H$, $z = c^{-1}b^{-1}a^{-1}cba = c^{-1}b^{-1}cbw \in H$, $r = c^{-1}b^{-1}a^{-1}cg(a, b) = za^{-1}b^{-1}g(a, b) = zh(b, a)^{-1} \in H$, $s = c^{-1}h(a, b)^{-1}c = rg(a, b)^{-1}c^{-1}g(a, b)c \in H$, and finally $t = s^{-1} = c^{-1}h(a, b)c \in H$.

Thus we have shown that $c^{-1}h(a, b)c \in H$; hence $h(a, b) \in L_G(H)$, $h(a, b) = 1$, and $g(a, b) = ab$. We conclude that A is a subsemigroup of G. If $a \in A$, then $a^{-1}H = bH$ for some $b \in A$. Now $H = abH$, hence $ab \in H$. Since A is H-selfconnected, $ab = 1$ and $a^{-1} = b \in A$. This means that A is a subgroup and since $abH = baH$ for all $a, b \in A$ we get $ab = ba$. The proof is complete.

EXAMPLE 3.2. Consider the alternating group A_4. Let H be a Sylow 3-subgroup (there are four Sylow 3-subgroups in A_4). Now H is cyclic and $L_G(H) = 1$. If A and B are H-connected transversals, then $A = B$ is the elementary abelian Sylow 2-subgroup.

COROLLARY 3.3. *Under the assumptions of Theorem* 3.1, *if* $G = \langle A, B \rangle$, *then* $G' = 1$.

LEMMA 3.4. *Let* $H \leqslant G$, *H cyclic, and* $L_G(H) = 1$. *Let A and B be left transversals to H in G and* $ab = ba$ *for every* $a \in A$ *and* $b \in B$. *If* $G = \langle A, B \rangle$, *then* $G = A = B$ *is abelian.*

Proof. Clearly, $C = \langle A \rangle \cap H$ is normal in G, hence $C = 1$ and $\langle A \rangle = A$ is a normal subgroup of G. Similarly, $\langle B \rangle = B$ is normal in G and we have $G = AB$. Thus $H = \langle ab \rangle$, where $a \in A$ and $b \in B$. Now $G = AH = A\langle b \rangle$. If $c \in B$, then $c = db^n$, where $d \in A$ and n is an integer. Since $d \in A \cap B$, it follows that $d \in Z(G)$. We conclude that B is abelian, $H = 1$, and $G = A = B$.

THEOREM 3.5. *Let H be a cyclic subgroup of a finite group G. Then* $G' \leqslant H$ *if and only if there exists a pair A, B of H-connected left transversals in G such that* $G = \langle A, B \rangle$.

Proof. Let A be any left transversal to H and h a generator of H. Then put $B = Ah$ and the direct implication is clear. We now prove the inverse implication. Assume that G is a counterexample of the smallest possible order.

If H is normal in G, then G/H is abelian and $G' \leqslant H$, a contradiction. If $1 < K < H$ and K is normal in G, then G/K satisfies our conditions and $(G/K)' \leqslant H/K$ yielding $G' \leqslant H$. We conclude that $L_G(H) = 1$. Furthermore, if $1 \neq N$ is a normal subgroup of G, then $G/L_G(HN)$ satisfies our conditions (see Lemma 2.8); hence $G' \leqslant L_G(HN)$ and HN is normal in G.

We now prove that $Z(G)=1$. If $z \in Z(G)$ $(z \neq 1)$, then $H\langle z\rangle$ is normal in G. If $x, y \in Z(G)$ $(x \neq 1, y \neq 1)$, x has order p, and y has order q (here p and q are two different prime numbers), then $H\langle x\rangle \cap H\langle y\rangle = H$. This means that H is normal in G which is not possible. It follows that $Z(G)$ is a p-group for a prime number p. Now $HZ(G)$ is normal in G and if Q is a Sylow q-subgroup of H $(q \neq p)$, then Q is a Sylow q-subgroup of $HZ(G)$ and consequently Q is normal in G. This means that H is a cyclic p-group. By Theorem 3.1, G is abelian which is a contradiction. We conclude that $Z(G)=1$.

By Proposition 2.7, we have that $N_G(H)=H=C_G(H)$. Let $H=P_1 \cdots P_r$, where the subgroups P_i are the Sylow p_i-subgroups of H. If $N_G(P_i)=H$ for every i then all the P_i's are Sylow subgroups of G satisfying $N_G(P_i)=C_G(P_i)$. By the theorem of Burnside (see [13, p. 419]) we know that each P_i has a normal complement in G and thus $G=KH$, where K is normal in G and $K \cap H=1$. Now clearly, $G' \leqslant K$ and since $a^{-1}b^{-1}ab \in H \cap G'$, we have $ab=ba$ for every $a \in A$ and $b \in B$. By Lemma 3.4, G is abelian and this is again a contradiction.

Thus we may assume that H has a Sylow p-subgroup P such that $N_G(P)>H$. Let T be minimal among those subgroups of $N_G(P)$ which properly contain H. Then H is a maximal subgroup of T and clearly $N_T(H)=H$ and $N_T(P)=T$. Since $C_T(P)$ is normal in $N_T(P)$, it follows that $C_T(P)>H$ and then $C_T(P)=T$, which yields $P \leqslant Z(T)$. Since $N_G(H)=H$, we have $Z(T)<H$. Now $Z(T)$ is a characteristic subgroup of T. If T is normal in G, then $Z(T)$ is normal in G but this contradicts $L_G(H)=1$.

Thus T is not normal in G. Now take an element $a \in (A \cap T)-H$. Then $T=\langle H, a\rangle$ and by Lemma 2.5, $a \in L_G(T)$. Thus $L=L_G(T) \neq 1$ and then $HL=T$ is normal in G, a contradiction. The proof is complete.

Example 3.6. Let G be the subgroup of S_6 generated by $A=\{e, (12)(34)(56), (135)(246), (164)(253), (145236), (154263)\}$ and let H be the stabilizer of 1 in G. Now A is H-selfconnected, G is of order 24, H is elementary abelian of order 4, and $N_G(H)$ is elementary abelian of order 8. Clearly, G' is not contained in H.

Corollary 3.6. *Let $1 \neq H$ be a cyclic subgroup of a finite group G with $L_G(H)=1$. If A, B is a pair of H-connected transversals in G, then $G \neq \langle A, B\rangle$.*

Remark. It seems to be an open problem whether the preceding result remains true in the infinite case. By Theorem 3.1, the answer is positive if H is a p-group or isomorphic to the Prüfer-group.

4. Quasigroups and Loops

Let Q be a *quasigroup* (i.e., a groupoid with unique division). For each $a \in Q$ we have two permutations L_a (left translation) and R_a (right translation) on Q defined by $L_a(x) = ax$ and $R_a(x) = xa$ for every $x \in Q$. The subgroup of S_Q generated by the set of all left and right translations is called the *multiplication group* of Q and is denoted by $M(Q)$. It is clear that $M(Q)$ is transitive on Q and the stabilizers of elements of Q are conjugated in $M(Q)$. If Q is a *loop* (i.e., a quasigroup with a neutral element e), then we denote the stabilizer of e by $I(Q)$ and we say that $I(Q)$ is the *inner mapping group* of Q. The concept of multiplication groups was introduced by Albert in [1, 2] and Bruck [6] introduced the notion of the inner mapping group (the analogue for loops of the inner automorphism group of a group).

We know that $I(Q) = 1$ if and only if Q is an abelian group. If Q is a group, then $I(Q)$ consists of the inner automorphisms of Q. In general, the inner mapping group is not a group of automorphisms. However, there are loops whose inner mapping group is a group of automorphisms: for example, the commutative Moufang loops (i.e., loops which satisfy the law $xx \cdot yz = xy \cdot xz$). These and many other results about inner mapping groups can be found in [8, 17, 19].

Now assume that Q is a loop and put $A = \{L_a \mid a \in Q\}$ and $B = \{R_a \mid a \in Q\}$. It is easy to see that A and B are $I(Q)$-connected (hence stable by Lemma 2.2) transversals to $I(Q)$ in $M(Q)$. Furthermore, if $1 < K \leqslant I(Q)$, then K is not normal in $M(Q)$. Finally, it is clear that $M(Q) = \langle A, B \rangle$.

We are now ready to state a theorem in which we reformulate and generalize some results from [3], [15] and [16].

Theorem 4.1. *A group G is isomorphic to the multiplication group of a loop if and only if there exists a subgroup H satisfying $L_G(H) = 1$ and H-connected transversals A and B satisfying $G = \langle A, B \rangle$.*

Proof. If Q is a loop then we can choose $G = M(Q)$, $H = I(Q)$, and A and B as in the discussion before the theorem.

Now assume that G has a subgroup H and H-connected transversals A and B satisfying the conditions of the theorem. For each $x \in G$, there is exactly one $f(x) \in A$ such that $f(x)H = xH$, i.e., $x^{-1}f(x) \in H$. Let K be the set of left cosets of H in G. We now define a binary operation $(*)$ on K by

$$(xH) * (yH) = f(x)\, yH.$$

If $u^{-1}x \in H$ and $v^{-1}y \in H$, then $f(x) = f(u)$ and $(f(x)v)^{-1} f(x)\, y = v^{-1}y \in H$. We conclude that $(*)$ is well-dfined. Now A is a stable transversal to H in G and by using this fact it is easy to see that $(K, *)$ is a quasigroup. Since A and B are H-connected and $L_G(H)=1$, it follows that $1 \in A$; hence $(K, *)$ is a loop. For each $x \in G$, there also exists exactly one $g(x) \in B$ with $xH = g(x)H$. Now $(xH) * (yH) = f(x)\,yH = f(x)\,g(y)H$. Since A and B are H-connected, we have $f(x)\,g(y)H = g(y)\,f(x)H$.

Then consider the action of G on K by left multiplication. Since $L_G(H)=1$, the kernel of the permutation representation corresponding to the action is trivial. We conclude that $M(K)$ is isomorphic to G, since $G = \langle A, B \rangle$. The proof is complete.

We also have

COROLLARY 4.2. *A group G is isomorphic to the multiplication group of a commutative loop if and only if there exists a subgroup H of G satisfying $L_G(H)=1$ and an H-selfconnected transversal A satisfying $G = \langle A \rangle$.*

We are now ready to prove

THEOREM 4.3. *Let Q be a loop such that $I(Q)$ is a cyclic group. Then Q is an abelian group provided that at least one of the following conditions is satisfied:*

(1) *Q is finite,*

(2) *Q is a group,*

(3) *$I(Q)$ is a p-group for a prime p.*

Hence in all these cases $I(Q)=1$.

Proof. Now (1) follows from Theorems 3.5 and 4.1. Clearly, (2) is folklore and (3) follows from Theorem 3.1.

Finally, we make some remarks about the preceding theorem and the situation in quasigroups.

Remark 1. The problem whether $I(Q)$ cyclic implies Q to be an abelian group is still unsolved in the general case.

Remark 2. Let $n \geqslant 2$ and consider the elementary abelian 2-group $(Q, +)$ of order 2^n and with basis $\{a_1, ..., a_n\}$. Let f be an automorphism of $(Q, +)$ such that $f(a_1) = a_2, ..., f(a_{n-1}) = a_n$ and $f(a_n) = a_1$. If we put $x * y = f(x+y)$ for all $x, y \in Q$, then $(Q, *)$ is a quasigroup (which is not a group) and the stabilizers of elements of Q in $M(Q)$ are cyclic groups of order n. By considering an elementary abelian 2-group with an infinite countable basis a similar construction gives an example of a quasigroup which has the property that the stabilizers of elements of Q are infinite cyclic groups.

Remark 3. Quasigroups whose multiplication groups have small stabilizers are described in [17, 18].

5. Multiplication Groups of Loops

All abelian groups are naturally isomorphic to multiplication groups of loops. For every $n \geqslant 5$ there exists a loop Q of order n such that $M(Q) = S_n$ (see [9, Theorem 3.1.1]). For every $n \geqslant 6$ there exists a loop Q of order n such that $M(Q) = A_n$ (see [11]).

Now consider a loop Q with the following multiplication table:

1	2	3	4	5	6
2	1	4	3	6	5
3	4	5	6	2	1
4	3	6	5	1	2
5	6	1	2	3	4
6	5	2	1	4	3

Here $M(Q)$ is a nonnilpotent group of order 24 and $I(Q)$ is isomorphic to Klein's four group.

The rest of the section is devoted to groups which are not multiplication groups of loops. Let G be a group and $1 < H < G$. Now we consider the following conditions on H:

(1) $L_G(H) > 1$,

(2) $N_G(H) > HZ(G)$,

(3) H is a cyclic p-group or H is isomorphic to the Prüfer group,

(4) H is cyclic.

Proposition 2.7, Theorem 4.1, and Theorem 4.3 yield

THEOREM 5.1. *Suppose that G is a group and each proper nontrivial subgroup of G satisfies either* (1), (2), *or* (3). *Then G is not isomorphic to the multiplication group of a loop. If G is a finite group and every proper nontrivial subgroup of G satisfies either* (1), (2), *or* (4) *then G is not isomorphic to the multiplication group of a loop.*

By using the preceding theorem it is easy to see that S_3, S_4, and A_4 are not isomorphic to multiplication groups of loops. The conditions of Theorem 5.1 are also satisfied by the following groups:

(1) Hamiltonian groups,

(2) Heineken–Mohamed groups (these infinite groups have trivial centre but the proper subgroups are properly contained in their normalizers; see, e.g., [12]),

(3) Blackburn groups (A finite group is a Blackburn group if it is neither abelian nor hamiltonian and the intersection of all nonnormal subgroups is not trivial; in a Blackburn group every minimal subgroup is normal; see [5]),

(4) Dihedral groups,

(5) Nonprimary Redei groups (a finite group is nonprimary Redei if it is not abelian, not of prime power order, and each of its subgroups is abelian; for the structure of these groups see [13, pp. 281 and 286]).

Remark 1. Drapal [10] has shown that A_5 is not isomorphic to the multiplication group of a loop. For dihedral groups this was first shown by Ihringer [15].

Remark 2. Hamiltonian groups and Heineken–Mohamed groups are not multiplication groups of quasigroups. This was first shown by Kepka [17] and Smith [20].

Remark 3. All finite dihedral, symmetric, alternating, general linear, projective general linear groups, and the Mathieu groups M_{11} and M_{23} are isomorphic to multiplication groups of quasigroups; see Ihringer [14].

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